FOUR-STEP BLOCK HYBRID EXPLICIT METHODS FOR THE SOLUTION OF INITIAL VALUE PROBLEMS

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Abstract: In this research paper, the continuous Block Hybrid explicit methods for k=4 was derived from Adams Bashforth methods. The continuous scheme was evaluated at different points to obtain discrete schemes. The order, error constant, zero stability and consistency of the resulting discrete schemes were ascertained. The region of absolute stability of the block hybrid scheme was plotted. The schemes were experimented in solving some stiff and non stiff initial value problems (IVP) in block form. It was observed that the block hybrid explicit methods obtained in this work performed better than the conventional Adams Bashforth method in terms of Accuracy, Efficiency and Stability.

Keywords: Explicit methods, stiff ODEs, region of absolute stability

Introduction

The limitation of analytical means in finding an exact solution to most modeled equations cannot be over emphasized. It then becomes necessary to apply numerical methods when faced with such problems. By numerical method, we mean a difference equation involving the number of consecutive approximation $y_{n+j}, j = 0, 1, \ldots k$, from which it will be possible to compute sequentially the sequence $y_n/n = 0, 1, \ldots N;$ the k is called the step number of the method. These numerical methods use the available discrete numerical integration algorithms in which the numerical approximations are obtained at some specific points in the interval of integration.

Initial value problems (IVP) is defined as any differential equation of the form

$$\left.\begin{aligned} y'(x) &= f(x\ y(x)), \\ y(x) &= y_0, a \le x \le b \end{aligned}\right\} \qquad (1.1)$$

on a given mesh,

$a = x_0 < x_2 < \ldots\ldots < N = b$ With a mesh size

The numerical solution of (1.1) is a major focus of this paper.

The linear multistep method (LMM), though efficient with regards to accuracy for a given number of functions evaluation per step, suffer the pitfall of poor stability property as step number increases. However, it was observed that some of the difficulties inherent in the linear multistep -methods can be reduced by lowering the step number and increasing the order without reducing the stability interval. This gives rise to the idea of a hybrid scheme. It is called hybrid because it possess some properties of LMM and that of Runge-Kutta methods (Stells & Gragg in Butcher & Burrage, 2004).

We therefore define K-hybrid scheme as follows:

$$\sum \alpha j \quad y_{n+j} = h \sum_{j=0}^{k} \sum_{j=0}^{k} \beta j \quad f_{n+j} + h\beta_v f_{n+v}$$

where $\alpha k = 1$, α_0 and B_0 are not both zero and V € {0, 1… k} and $f_{n+v} = f(k_{n+v}, y_{n+v})$ which is the off-grid function evaluation. One main disadvantage of the hybrid scheme is that they require special predictor to predict the off-grid points. In this paper, this problem is solved with the use of block method.

The block method is purposed to ease computational effort and prevent the use of starting values for the hybrid discrete schemes. These schemes are obtained by evaluating the various continuous schemes at both grid and off-grid points as the case may be, which are then simplified to obtain the block method of each scheme.

Derivation Techniques

Consider the initial value problem (1.1)

$$y' = f(x, y); a \le x \le b, y(a) = yo,$$

on a given mesh $a = xo < x_1 < x_2 < \ldots < x_m = b$,

where $h = x_{n+1} - x_n, n = 0,1,n,$ where h is a constant step and k is the step number of the method.

In order to solve equation (1.1) Onumanyi *et al.* (1994) developed a linear multistep method with continuous coefficient by the idea of multistep collocations.

Let $y(x) = \sum_{j=0}^{t-1} \alpha j(x) y(x_{n+i}) + h \sum \beta(x) f(\bar{x}j, \bar{y}(xj))$--- (2.1)

where $\alpha j(x) = \sum_{i=0}^{k+m-1} \alpha j,\ i+1\, x^i$

$$h\beta j(x) = \sum_{i=0}^{k+m-1} h\beta j, i+1 x^i$$

Also
$$y(x_{n+j}) = jn + j,\ j \in (0,1,\ldots k = 1)$$
$$y1(\bar{x}j) = f(xj, y(\bar{x}j)),\ j = 1,\ldots\ldots m$$

to get $\alpha j(x)$ and $\beta_j^{(x)}$, Sirisena [2003] used the matrix equation of the form

DC=I ----------------------------------- (2.2)

where I is the identity matrix of dimension (t+m) x (t+m) while D and C are matrices defined as:

$$D=\begin{pmatrix} 1, x_n, x_n^2, \ldots\ldots\ldots\ldots, x_n^{t+m-1} \\ 1, x_{n+1}, x^2{}_{n+1}, \ldots\ldots\ldots\ldots, x_{n+1}^{t+m-1} \\ \vdots \quad \vdots \\ \vdots \quad \vdots \\ \vdots \quad \vdots \\ 1, x_{n+t-1}, x^2{}_{n+t-1}, \ldots\ldots\ldots\ldots, x_{n+t-1}^{t+m-1} \\ 0, \quad 1, \quad 2x_0 \ldots\ldots\ldots, (t+m-1)x_0^{t+m-2} \\ \vdots \quad \vdots \\ \vdots \quad \vdots \\ \vdots \quad \vdots \\ 0, \quad 1, \quad 2x_{m-1} \ldots\ldots\ldots, x_{m-1}^{t+m-2} \end{pmatrix} \qquad \text{----- (2.3)}$$

The above matrix (2.3) is the multistep collocation matrix of dimension (t+m) x (t+m), and

$$C=\begin{pmatrix} \alpha_{0,1} & \alpha_{11}\,--- & \alpha t_{-1,1} & h\beta_{01} & h\beta_{m-1,1} \\ \alpha_{02} & \alpha_{12}\,--- & \alpha t_{-1,2} & h\beta_{02} & h\beta_{m-1,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1_{\alpha 0,t+m} & \alpha_{1,t+m} & \alpha_{t-1,t+m} & h\beta_{0t+m} & h\beta_{m-1,t+m} \end{pmatrix} \qquad \text{----- (2.4)}$$

We define t as the number of interpolation points used while m is the number of collocation points used. The columns of the matrix C $=D^{-1}$ gives the continuous coefficients $\alpha_j(x),\ j=0,1,k-1$ and $\beta_j(x),\ j=0,1,\ldots..\ k-1$

Block hybrid explicit methods of step number K=4

Consider the discrete Adams Bashforth method of order 2

$$y_{n+2}=y_{n+3}+\frac{h}{24}\left[55f_{n+3}-59f_{n+2}+37f_{n+1}-9f_n\right]\ldots(3.1)$$

Expressing (3.1) in the general form gives;

$$y(x)=\alpha_1^{(x)}y_{n+}+h\left[\beta o^{(x)}f_n+\beta_1^{(x)}f_{n+1}+\beta_1^{(x)}f_{n+\frac{3}{2}}+\beta_1^{(x)}f_{n+2}+\beta_1^{(x)}f_{n+2}+\beta_1^{(x)}f_{n+3}\right] \text{-------}\ldots\ldots\ldots3.2)$$

Introducing one off grid collocation points at $x=x_{n+3/2}$ and $x=x_{n+5/2}$ in x gives

$$y(x)=\alpha 1(x)yn+1+h\left[\beta o^{(x)}f_n+\beta_{3/4}(x)f_{n+3/2}+\beta_1^{(x)}f_{n+1}+\beta_2^{(x)}f_{n+2}+\beta_{3/4}(x)f_{n+5/2}+\beta_3^{(x)}f_{n+3}\right] \text{-- (3.3)}$$

and the resultant matrix is given below as,

$$D=\begin{pmatrix} 1 & x_n+2h & (x_n+2h)^2 & (x_n+2h)^3 & (x_n+2h)^4 & (x_n+2h)^5 \\ 0 & 1 & 2x_n & 3x_n^{\ 2} & 4x_n^{\ 3} & 5x_n^{\ 4} \\ 0 & 1 & 2\left(x_n+\frac{3}{2}h\right)^2 & 3\left(x_n+\frac{3}{2}h\right) & 4\left(x_n+\frac{3}{2}h\right)^3 & 5\left(x_n+\frac{3}{2}h\right)^4 \\ 0 & 1 & 2(x_n+2h)^2 & 3(x_n+2h) & 4(x_n+2h)^3 & 5(x_n+2h)^4 \\ 0 & 1 & 2\left(x_n+\frac{5}{2}h\right)^2 & 3\left(x_n+\frac{5}{2}h\right) & 4\left(x_n+\frac{5}{2}h\right)^3 & 5\left(x_n+\frac{5}{2}h\right)^4 \\ 0 & 1 & 2(x_n+3h)^2 & 3(x_n+3h) & 4(x_n+3h)^3 & 5(x_n+3h) \\ 0 & 1 & 2(x_n+4h)^2 & 3(x_n+4h) & 4(x_n+4h)^3 & 5(x_n+4h)^4 \end{pmatrix} \text{------- (3.4)}$$

The elements of matrix C are obtained from the inverse of D by the use of maple 7 software and after simplification we obtain the continuous form expressed as

$y(x)=$

$$y_{n+1} + \left(\frac{2}{9}^{(k+h)} \left(\frac{\frac{5}{4}h^2 + \frac{13}{4}kh + 2k^2}{h^2} \right) + \frac{4}{3}\left(\frac{\frac{3}{4}h^2 + \frac{7}{4}kh + k^2}{h^2} \right) x_{-\frac{2}{3}} \left(\frac{\frac{7}{4}k + 2k}{h^2} \right) x^2\, 9\frac{\frac{4}{x^3}}{h^2} \right) f_n$$

$$+ \left(\frac{8}{9}(k+h)^2 (2k-h)^2 (2k-h) - \frac{16}{3}\left(\frac{k+h}{h^2}\right) x_{+} \frac{\overset{8}{3}(h+2k)x^2}{h^2} - \frac{16}{9}\frac{x^3}{h^2} \right) f_{n+3/4} \qquad \text{--- (3.5)}$$

$$+ \left(-\frac{2}{3}^{(k+h)} \frac{\left(2k^2 + \frac{1}{4}kh - \frac{1}{4}h^2\right)}{h^2} + \frac{4\left(k + \frac{3}{4}h\right)kx}{h^2}\ \frac{2\left(2k + \frac{3}{4}h\right)x^2}{h^2} + \frac{\frac{4}{3}x^3}{h^2} \right) f_{n+1}$$

Evaluating (3.5) at $x = x_n, x_n + {}^3\!/_4, x_{n+1},\ x_{n+2} x = x_n, x_n + {}^3\!/_2, x_{n+31}$ and x_{n+4} yields the following discrete schemes as shown below;

Four-step block hybrid explicit methods (BHEM)

$$y_n := y_{n+1} - \frac{1}{1080} h \left(297 f_n + 2019 f_{n+1} - 2496 f_{n+\frac{3}{2}} - 768 f_{n+\frac{5}{2}} + 1899 f_{n+2} + 29 f_{n+3} \right)$$

$$y_{n+\frac{3}{2}} := y_{n+1} - \frac{1}{69120} h \left(108 f_n - 13668 f_{n+1} - 26688 f_{n+\frac{3}{2}} - 2496 f_{n+\frac{5}{2}} + 7812 f_{n+2} + 47 f_{n+3} \right)$$

$$y_{n+2} := y_{n+1} - \frac{1}{1080} h \left(f_n - 189 f_{n+1} - 704 f_{n+\frac{3}{2}} - 189 f_{n+2} + f_{n+3} \right)$$

$$y_{n+\frac{5}{2}} := y_{n+1} - \frac{1}{2560} h \left(4 f_n - 492 f_{n+1} - 1472 f_{n+\frac{3}{2}} - 576 f_{n+\frac{5}{2}} - 1332 f_{n+2} - 47 f_{n+3} \right)$$

$$y_{n+3} := y_{n+1} + \frac{1}{270} h \left(42 f_{n+1} + 192 f_{n+\frac{3}{2}} + 192 f_{n+\frac{5}{2}} + 72 f_{n+2} + 67 f_{n+3} \right)$$

…………………… (4.1)

The scheme is consistent and zero stable; hence it is convergent.
In summary, the block method has the following order and error constants

Table 1: Order and error constants for BHEM k=4

Evaluating Point	Order	Error constants
$X = x_n$	5	-0.0004122
$X = x_{n+\frac{3}{2}}$	5	-0.0002478
$X = x_{n+2}$	5	0.0112649
$X = x_{n+\frac{5}{2}}$	5	0.0006422
$X = x_{n+3}$	5	0.031542

The methods are all convergent since there are all consistent and zero-stable.

Stability regions of the block hybrid explicit methods

To plot the region of absolute stability of the block hybrid method, the newly constructed methods are reformulated as General Linear Methods and expressed as

$$\begin{bmatrix} y \\ y_{i+1} \end{bmatrix} = \left\langle \frac{A}{B} \middle| \frac{U}{V} \right\rangle \begin{bmatrix} \begin{pmatrix} hf(y) \\ y_{n+1} \end{pmatrix} \end{bmatrix} \qquad \text{------------------ (5.1)}$$

Where A= $\begin{bmatrix} a_1 & K & a_{1s} \\ a_s & K & a_{ss} \end{bmatrix}, \; B = \begin{bmatrix} b_{11} & K & b_{1s} \\ b_S & K & B_{SS} \end{bmatrix}$

The elements of u and v are obtained from the interpolation and collocation points respectively. The elemenst of the matrices A, B, U and V are substituted into the stability matrix

$$m(z) = B_2 + ZA_2(I - ZA_1)^{-1}B_1 \text{ ------------------ (5.2)}$$

and the stability function $P(\eta, z) = \det(\eta I - M(z))$------ (5.3)

$$Y = \begin{bmatrix} y_n \\ y_{n+1} \\ M \\ y_{n+k} \end{bmatrix}, \quad y_{i+1} = \begin{bmatrix} y_{n+k} \\ y_{n+k-1} \end{bmatrix}, \quad y_{i-1} = \begin{bmatrix} y_{n+k-1} \\ y_{n+k-2} \end{bmatrix} \text{..... (5.4)}$$

Using maple 7 software gives the stability polynomial below;

$$f(z) = -7\,(2055\ \eta\ z^2 + 500\ \eta\ z\ - 7616\ \eta + 35\ z^2 + 4935\ z + 5616)/(9\ z^2\ 43\ z\ + 94)^2$$

$$f(Z.P) = (\ -512\ z^2\ -375\ \eta\ z^2 + 7\ \eta^2 + 22\ \eta^2\ z\ -470\ z) + 279\ \eta\ z\ -255\eta + 96\ \eta^2 + 94) \quad \text{......... (5.5)}$$

Using MATLAB software and stability polynomial the stability region of the block method is plotted and is shown to be A-SATBLE (Fig. 1).

Region of absolute stability graph for block hybrid Adams Bashforth for k=4 with off grid points at $x = x_{n+3/2}$ and $x = x_{n+5/2}$

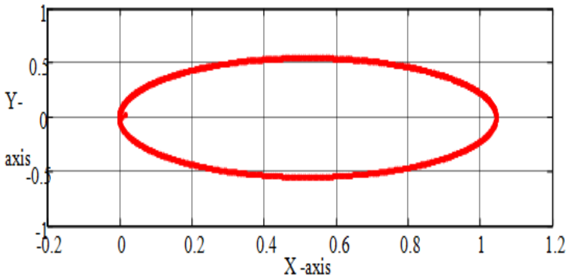

Fig. 1: Region of absolute stability of BHEM K=4

From the graph the BHEM for K=4 is A-stable

Numerical experiment

We report here a numerical example on stiff problem taken from the literature using the solution curve. In comparison, we also report the performance of the new blended block linear multistep methods and the well-known Matlab stiff ODE solver ODE15S on the same problems and on the same axes.

Problem 1: Euler's equation of motion for a rigid body without external forces

$$y_1' = y_2 y_3$$

$$y_2' = -y_1 y_2$$

$$y_3' = -5.1 y_1 y_2$$

$$y_1(0) = 0, y_2(0) = 1, y_3(0) = 1$$

$$0 \le x \le 10, \quad h = 0.1$$

Problem 2: Linear stiff system of ODE

$$y_1' = -500000.5\, y_1 + 4.99999.5\, y_2$$

$$y_2' = 499999.5 y_1 - 500000.5\, y_2$$

$$y_1(0) = 0, y_2(0) = 2 \qquad 0 \le x \le 100, \quad h = 0.1$$

Theoretical solution is given by;

$$y_1(t) = -e^{t\lambda_1} + e^{t\lambda_2}, \qquad y_2(t) = e^{t\lambda_1} - e^{t\lambda_2}$$

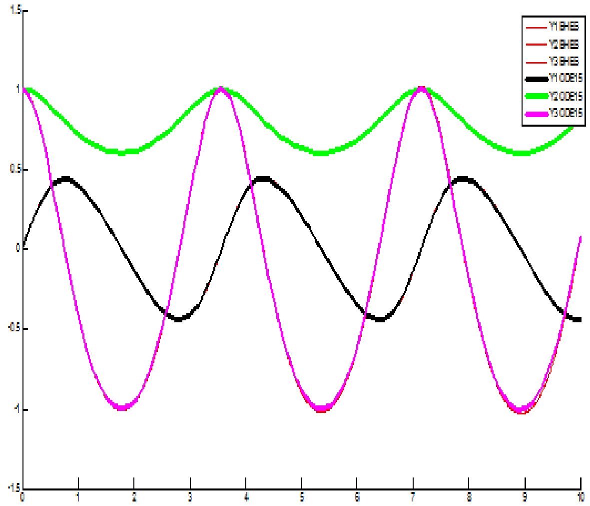

Fig. 2: Solution curve for Problem 1

Table 2: Absolute errors of the first component for Problem 2 (k=4)

h=0.1	HLMM 4 Error
0	0.0000E+00
0.1	5.9851E-04
0.2	2.9876E-04
0.3	2.9824E-04
0.4	2.9761E-04
0.5	2.9697E-04
0.6	2.9634E-06
0.7	2.9571E-06
0.8	2.9508E-08
0.9	2.9445E-10
1.0	2.9382E-10
1.1	2.9320E-10
1.2	2.9257E-12
1.3	2.9195E-12
1.4	2.9133E-12
1.5	2.9071E-12
1.6	2.9009E-14
1.7	2.8947E-14
1.8	2.8885E-14
1.9	2.8823E-16
2.0	2.8823E-16

The solutions curve in Fig. 2 shows that our method BHEM compete favourably with the ODE solver 15s. From problem 2, Table 2 shows that our methods performed well with marginal absolute error constants. From the stiff ode problems solved, BHEM tend to converge much faster to the theoretical solution. The method is therefore recommended for the solutions of mildly initial value problem.

Conclusion

A block hybrid explicit method has been constructed through the multistep collocation approach. The Region of absolute stability of the explicit BHEMs has been greatly enhanced (A-stable). These methods are all convergent. Numerical results reveal that the BHEMs tend to converge much faster to the theoretical solution in problem 2 in table 2 and the solution curve in problem 1 which shows the efficiency of the methods in solving stiff systems despite the fact that they are explicit methods.

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